Companion note to Hack, Istrefi, and Meier (2023):

A Local Projection Representation of State-Dependencies in DSGE Models

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Abstract

For a broad class of macroeconomic models, we show that the approximate equilibrium dynamics can be represented by state-dependent local projections. Our results provide a useful guide for the empirical analysis of state-dependencies in the propagation of macroeconomic shocks.

Section 1 introduces a class of non-linear DSGE models. Section 2 provides conditions under which the state-dependent local projection representation arises from these models. Finally, Section 3 exemplifies the general result using the New Keynesian model in Hack et al. (2023) and an extension.

1 A class of models

We first introduce a class of macroeconomic models, followed by a brief discussion of the generality of this class of models.

We consider the class of models in which the equilibrium conditions satisfy

$$\tilde{A}(\phi_t) z_t = \tilde{F}(\phi_t)\, \mathbb{E}_t z_{t+1} + \tilde{G}(\phi_t)\, x_t + \tilde{H}(\phi_t)\, z_{t-1}, \tag{1}$$

where $z_t \in \mathbb{R}^{N \times 1}$ denotes a vector of endogenous variables, $x_t \in \mathbb{R}^{K \times 1}$ denotes a vector of exogenous variables, and $\phi_t \in \mathbb{R}$ denotes a *state variable*. It captures a time-varying model parameter (e.g., systematic monetary policy) that affects the equilibrium dynamics of the model. In the absence of fluctuations in ϕ_t, the model is linear. The expectation operator, $\mathbb{E}_t$, conditions on the information set at time t consisting of $\{z_{t-i}, x_{t-i}, \phi_{t-i}\}_{i=0}^{\infty}$. The coefficient matrices, $\tilde{A}(\phi_t)$, $\tilde{F}(\phi_t)$, $\tilde{G}(\phi_t)$, $\tilde{H}(\phi_t)$, have conforming dimensions and are continuously differentiable in ϕ_t.

The vector x_t follows a stable VAR(1) process

$$x_t = P x_{t-1} + \varepsilon_t, \quad \varepsilon_t \overset{iid}{\sim} (\mathbf{0}, \Sigma_\varepsilon), \tag{2}$$

that is, all eigenvalues of $P \in \mathbb{R}^{K \times K}$ are strictly inside the unit circle. This formulation is general insofar as any stable AR(p) with p finite can be recast as a VAR(1) via the companion form representation (e.g., Hamilton, 1994). Finally, $\varepsilon_t \in \mathbb{R}^{K \times 1}$ denotes a vector of macroeconomic shocks that are mutually uncorrelated, i.e., $\Sigma_\varepsilon = \text{diag}(\sigma_1^2, \dots, \sigma_K^2)$.

The state variable ϕ_t follows a stable process

$$\phi_t = \rho_\phi \phi_{t-1} + \zeta_x \varepsilon_t + \eta_t, \quad \eta_t \overset{iid}{\sim} (0, \sigma_\eta), \quad \mathbb{E}[\varepsilon_t \eta_t] = \mathbf{0}, \tag{3}$$

that is, $|\rho_\phi| < 1$. The state variable may respond to the macroeconomic shocks ε_t, as governed by $\zeta_x \in \mathbb{R}^{1 \times K}$, but it may also fluctuate in response to *state shocks* η_t that are uncorrelated with ε_t. We restrict ϕ_t to follow a first-order autoregressive process with the goal of keeping the exposition more transparent. However, our results can be generalized to stable AR(p) processes with finite p.

Discussion. The class of models described by (1)–(3) is quite general insofar as we allow for forward- and backward-looking z_t to enter the system. It nests models with and without endogenous persistence (e.g., Clarida, Gali, and Gertler, 1999). The class also nests larger scale models (e.g., Smets and Wouters, 2007), and heterogeneous agent economies approximated by a finite state-space (e.g., Auclert, Bardóczy, Rognlie, and Straub, 2021). All of the quoted papers, however, impose time-invariant parameters, in which ϕ_t is a constant, an assumption which we relax. A model with time-varying parameters that falls into the class of models is the stylized New Keynesian model in Hack et al. (2023) in which ϕ_t corresponds to fluctuations in systematic monetary policy, in particular the inflation coefficient of the policy reaction function. The class of models does not nest models with state variables following a discrete Markov switching process (e.g., Bianchi, 2013), unless we consider the limit with continuous states.

2 Model solution

We next state two assumptions that permit representing the approximate model dynamics for the class of models introduced above as a state-dependent local projection. We then show how to obtain this model representation and provide conditions for the existence and uniqueness of a solution.

While our class of models allows for a non-linearity with regard to ϕ_t, it nests a class of linear DSGE models when $\phi_t = 0,\ \forall t$. First, we assume that the nested linear models have a unique bounded rational expectations solution:

Assumption 1 (**Blanchard-Kahn conditions**). *If $\phi_t = 0,\ \forall t$, then the model in* (1)–(2) *satisfies the Blanchard and Kahn (1980) conditions and admits a unique bounded rational expectations solution.*

This assumption will allow us to provide analytical conditions for the existence and uniqueness of a state-dependent local projection solution in our class of non-linear models, when departing from a well-understood class of linear models.

Second, we assume local invertibility of $\tilde{A}(\phi_t)$:

Assumption 2 (**Local invertibility**). *Matrix $\tilde{A}(\phi_t)$ is invertible $\forall \phi_t \in (-\delta, \delta)$ and $\delta > 0$.*

This assumption is frequently satisfied or can be satisfied by substituting out variables and equations that cause the rank deficiency of $\tilde{A}(\phi_t)$. Section 3 provides an example.

Approximate equilibrium conditions. We next derive the approximate non-linear equilibrium conditions in two steps. First, we use Assumption 2 to rewrite (1) as

$$z_t = F(\phi_t)\,\mathbb{E}_t z_{t+1} + G(\phi_t)\,x_t + H(\phi_t)\,z_{t-1}, \tag{4}$$

for $|\phi_t| < \delta$ and some $\delta > 0$, and with conformingly defined matrices $F(\phi_t)$, $G(\phi_t)$, and $H(\phi_t)$. Second, we take a first-order Taylor approximation of equation (4) with respect to ϕ_t at $\phi_t = 0$, to obtain the approximate equilibrium conditions

$$\begin{aligned} z_t = &F_0 \, \mathbb{E}_t z_{t+1}\Big|_{\phi_t=0} + G_0 x_t + H_0 z_{t-1} \\ &+ \left(F_1 \, \mathbb{E}_t z_{t+1}\Big|_{\phi_t=0} + F_0 \, \mathbb{E}_t \partial_{\phi_t} z_{t+1}\Big|_{\phi_t=0} + G_1 x_t + H_1 z_{t-1} \right) \phi_t + \mathcal{O}(\phi_t^2), \end{aligned} \tag{5}$$

where we use the notation that $F(0) = F_0,\ G(0) = G_0,\ H(0) = H_0$, and $F'(0) = F_1,\ G'(0) = G_1,\ H'(0) = H_1$, with F' denoting the element-wise partial derivative with respect to ϕ_t, and similar for G' and H'. The approximation error is of second order, $\mathcal{O}(\phi_t^2)$, and dropped in the subsequent analysis. If the second row of equation (5) is zero, this corresponds to a linearized model with constant parameter $\phi_t = 0$, as commonly studied in the literature. This nested linear model has a unique bounded rational expectations solution due to Assumption 1. If the second row is non-zero, then (5) captures the model dynamics with time-varying ϕ_t around the point of approximation, $\phi_t = 0$.

Perceived law of motion. To deriver the equilibrium dynamics of z_t, we next postulate the perceived law of motion (LoM) for z_t:

$$z_t = a + B_x x_t + B_z z_{t-1} + C_x x_t \phi_t + C_z z_{t-1} \phi_t + d\, \phi_t, \tag{6}$$

with unknown coefficient matrices $a, d \in \mathbb{R}^{N\times 1}$, $B_x, C_x \in \mathbb{R}^{N\times K}$, and $B_z, C_z \in \mathbb{R}^{N\times N}$. The next proposition shows that the perceived law of motion constitutes a fixed point of (5). In other words, the actual law of motion in the rational expectations equilibrium conforms with the perceived law of motion.

Proposition 1 (**Equilibrium dynamics**). *Given the approximate equilibrium conditions in* (5) *and the perceived law of motion in* (6)*, the implied actual law of motion conforms with the perceived law of motion, provided that a solution exists. Hence, the equilibrium dynamics of z_t are parametrically described by* (6)*.*

Proof: In Appendix A.1.

Several further insights follow from this result and the associated proof. A first corollary is that the equilibrium dynamics have an equivalent representation as a state-dependent local projection. Without loss of generality, we focus on a single shock ε_{1t}, the first row of ε_t.

We take the model solution for the vector z_{t+h} and iterate it backward to obtain the state-dependent local projection representation.

Corollary 1 (**State-dependent local projection**). *For any $h \geq 0$, the equilibrium dynamics for z_t can be represented by*

$$z_{t+h} = \alpha^h + \beta^h \varepsilon_{1t} + \gamma^h \varepsilon_{1t}\phi_t + \delta^h \phi_t + u_{t+h}, \tag{7}$$

where the coefficients are defined recursively

$$\alpha^h = a + B_z \alpha^{h-1}, \tag{8}$$
$$\beta^h = B_x P^h \iota_1 + B_z \beta^{h-1}, \tag{9}$$
$$\gamma^h = C_x P^h \iota_1 \rho_\phi^h + B_z \gamma^{h-1} + C_z \rho_\phi^h \beta^{h-1}, \tag{10}$$
$$\delta^h = d\rho_\phi^h + B_z \delta^{h-1} + C_z \rho_\phi^h \alpha^{h-1},, \tag{11}$$

with $\iota_1 \in \mathbb{R}^{K\times 1}$ being the first unit vector. The initial conditions are $\alpha^0 = a$, $\beta^0 = B_x \iota_1$, $\gamma^0 = C_x \iota_1$, $\delta^0 = d$, and and u_{t+h} is a conformingly defined error term.

Proof: In Appendix A.2.

To obtain the coefficients of the local projection, we iterate (7) backwards and collect all terms that load on the right-hand side variables. For example, the intercept α^h is simply the constant from the LoM plus an extra term. The extra term emerges from substituting the y_{t+h-1} on the right-hand side, which again, features an intercept α^{h-1}. The same holds true in the remaining equations (9)-(11): in each of these equations, there are several additional terms that enter due to the backward-looking component.
The proof of Proposition 1 also delivers equations that characterize the coefficient matrices of (6) such that the perceived and actual law of motion coincide. We use these equations to obtain necessary and sufficient conditions for the existence of a unique solution consistent with the perceived LoM.

Corollary 2 (**Existence and uniqueness**). *Under Assumptions 1 and 2, a unique solution to* (5) *that is consistent with the perceived LoM in* (6) *exists if and only if matrices*

$\mathcal{M}_{C_z}$, $\mathcal{M}_{C_x}$, $\mathcal{M}_a$, *and* $\mathcal{M}_d$ *are invertible, where*

$$\mathcal{M}_{C_z} = \left[\mathbb{I}_{N^2} - (\mathbb{I}_N \otimes F_0 B_z) - (B_z^T \otimes F_0)\rho_\phi\right], \tag{12}$$

$$\mathcal{M}_{C_x} = \left[\mathbb{I}_{NK} - (\mathbb{I}_K \otimes F_0 B_z) - (P^T \otimes F_0)\rho_\phi\right], \tag{13}$$

$$\mathcal{M}_a = \left[\mathbb{I}_N - F_0 - F_0 B_z\right], \tag{14}$$

$$\mathcal{M}_d = \left[\mathbb{I}_N - F_0 B_z - F_0 \rho_\phi\right], \tag{15}$$

and $\mathbb{I}_M \in \mathbb{R}^{M\times M}$ *denotes the identity matrix for all* $M > 0$ *and* T *the transpose operator.*

Proof: In Appendix A.3.

Intuitively, once the perceived LoM is imposed in (5), these conditions determine that we can invert several systems of linear equations to obtain an explicit expression for the coefficient matrices of the perceived LoM. There is a similar matrix $\mathcal{M}_{B_x}$ that needs to be invertible, and a second-order matrix polynomial equation for B_z to be solved. Assumption 1 ensures invertibility and enables us to pick the correct root of the matrix polynomial equation. The reason is that the equations characterizing B_z and B_x coincide with the corresponding equations that arise from the nested linear model in (5) when $\phi_t = 0, \ \forall t$. Provided that the above invertibility conditions hold, and given B_z and B_x, we obtain closed-form expressions for all remaining coefficient matrices and state these formulas in the last corollary.

Corollary 3 (Coefficients). *Under the conditions stated in Corollary 2 and given* B_z *and* B_x*, the unique solution to* (5) *that satisfies the perceived LoM in* (6) *is given by the coefficient matrices*

$$vec(C_z) = \mathcal{M}_{C_z}^{-1}\, vec(F_1 B_z^2 + H_1) \tag{16}$$

$$vec(C_x) = \mathcal{M}_{C_x}^{-1}\, vec(F_1 B_x P + F_1 B_z B_x + F_0 C_z B_x \rho_\phi + G_1) \tag{17}$$

$$a = \mathcal{M}_a^{-1} F_0 C_x \Sigma_\varepsilon \zeta_x' \tag{18}$$

$$d = \mathcal{M}_d^{-1}\left(F_1 a + F_1 B_z a + F_1 C_x \Sigma_\varepsilon \zeta_x' + F_0 C_z \rho_\phi a\right), \tag{19}$$

with $vec(M)$ *denoting the column-wise vectorization operator for any matrix* M*.*

Proof: In Appendix A.3.

Two remarks are in order. First, Assumption 1, the Blanchard-Kahn condition, ensures the existence and uniqueness of a bounded rational expectations solution for nested linear model. The same condition support existence and uniqueness in a local approximation around the linear model. Second, Corollaries 2 and 3 are useful from a practical point of view. Given a calibrated model, it is easy to verify existence and uniqueness, and the computation of the model solution does not require solving a fixed point problem beyond the familiar linear solution.

A sufficient condition. While the conditions in Corollary 2 are necessary and sufficient, whether or not they are satisfied may not be particularly easy to characterize analytically. Thus, we propose a sufficient condition for the existence and uniqueness of an equilibrium solution that relies on one assumption only, which is relatively easy to investigate analytically:

Assumption 3. *Matrix F_0 is invertible.*

Assumption 3 is not an unfamiliar one. For the nested linear model in (5), the assumption allows representing the equilibrium conditions in the so-called canonical form proposed in Blanchard and Kahn (1980). Given this assumption, we can prove that the conditions from Corollary 2 are satisfied, which implies the following result.

Proposition 2 (Sufficient condition for uniqueness). *Under Assumptions 1–3, a unique solution to (5) that satisfies the functional form of the perceived LoM in (6) exists, so that the coefficient matrices a, B_x, B_z, C_x, C_z, and d are uniquely determined.*

Proof: In Appendix A.4.

In the next section, we will illustrate our results in the context of a New Keynesian model featuring a time-varying state variable.

3 Illustrative examples

A simple model. To illustrate our results, we first consider the three-equation model as in Hack et al. (2023), which consists of a dynamic IS equation

$$y_t = \mathbb{E}_t y_{t+1} - (1-\gamma)(i_t - \mathbb{E}_t \pi_{t+1}) + \gamma(x_t^s - \mathbb{E}_t x_{t+1}^s), \tag{20}$$

and a New Keynesian Phillips curve

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa_0 y_t + \kappa_1 x_t^a - \lambda\gamma/(1-\gamma) x_t^s, \tag{21}$$

where y_t, π_t, and i_t denote real output, inflation, and nominal interest rates, respectively. The variables x_t^a, x_t^s denote exogenous productivity and government spending, which follow mutually independent and stable AR(1) processes. Lastly, we have a Taylor rule with time-varying systematic monetary policy,

$$i_t = (\phi + \phi_t)\pi_t, \tag{22}$$

with only a response to inflation, which varies over time according to ϕ_t. Substituting in the Taylor rule yields the following matrix representation corresponding to $z_t = (y_t, \pi_t)'$ and $x_t = (x_t^a, x_t^s)'$:

$$\tilde{A}(\phi_t) = \begin{bmatrix} 1 & (1-\gamma)(\phi+\phi_t) \\ -\kappa_0 & 1 \end{bmatrix}, \quad \tilde{F}(\phi_t) \quad = \begin{bmatrix} 1 & 1-\gamma \\ 0 & \beta \end{bmatrix}, \quad \tilde{G}(\phi_t) = \begin{bmatrix} 0 & \gamma(1-\rho_s) \\ \kappa_1 & -\frac{\lambda\gamma}{1-\gamma} \end{bmatrix}$$

It is easy to see that Assumption 2 is satisfied and we will take Assumption 1 as given. Second, to assess invertibility of F_0 (Assumption 3), we compute

$$F_0 = \frac{1}{1+\kappa_0(1-\gamma)\phi} \begin{bmatrix} 1 & (1-\gamma)(1-\beta\phi) \\ \kappa_0 & \kappa_0(1-\gamma)+\beta \end{bmatrix},$$

which implies $\det(F_0) = \frac{\beta}{1+\kappa_0(1-\gamma)\phi}$. Hence, Assumption 3 holds under the standard parameter restrictions. This implies the existence and uniqueness of a solution to our model, and an equivalent representation as a state-dependent local projection.

A model extension. The chosen model example is very simple. We deliberately present a simple model because adding backward-looking terms does not alter what we have derived thus far. Specifically, let us consider the model extension from Clarida et al. (1999), in which backward-looking terms in the Phillips curve and IS equation are introduced. In this case, these equations read

$$y_t = \mathbb{E}_t y_{t+1} - (1-\gamma)(i_t - \mathbb{E}_t\pi_{t+1}) + \gamma(x_t^s - \mathbb{E}_t x_{t+1}^s) + \chi_o o_{t-1} \tag{23}$$

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \kappa_0 y_t + \kappa_1 x_t^a - \lambda\gamma/(1-\gamma)x_t^s + \chi_\pi\pi_{t-1}, \tag{24}$$

where $\chi_o o_{t-1}$ and $\chi_\pi\pi_{t-1}$ are the newly added terms, and the related literature presents various microfoundations for these augmented aggregate relations. In this case, the above stated matrices remain valid, except that matrix $\tilde{H}(\phi_t)$ contains non-zero entries. This does not change any of the above derivations since they did not involve $\tilde{H}(\phi_t)$.

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Appendix A Derivations

A.1 Proof of Proposition 1

To establish the Proposition, we present a constructive proof that delivers an explicit solution, which also implies Corollaries 2 and 3.

Characterizing expectational terms. Using the perceived LoM (6), we compute all expectation terms from equation (5). We start with $\mathbb{E}_t z_{t+1}\big|_{\phi_t=0}$ and insert the perceived law of motion iteratively to obtain

$$
\begin{aligned}
\mathbb{E}_t z_{t+1}\Big|_{\phi_t=0} &= \mathbb{E}_t\{a + B_x x_{t+1} + B_z z_t + C_x x_{t+1}\phi_{t+1} + C_z z_t \phi_{t+1} + d\,\phi_{t+1}\}\Big|_{\phi_t=0} \\
&= \mathbb{E}_t\{a + B_x x_{t+1} + B_z\,[a + B_x x_t + B_z z_{t-1} + C_x x_t \phi_t + C_z z_{t-1}\phi_t + d\,\phi_t.] \\
&\qquad\qquad + C_z\,[a + B_x x_t + B_z z_{t-1} + C_x x_t \phi_t + C_z z_{t-1}\phi_t + d\,\phi_t.]\,\phi_{t+1} \\
&\qquad\qquad\quad + C_x x_{t+1}\phi_{t+1} + d\,\phi_{t+1}\}\Big|_{\phi_t=0}.
\end{aligned}
\tag{A.1}
$$

Equation (A.1) contains a cross-term associated with C_x that we characterize next.

$$
\begin{aligned}
\mathbb{E}_t[x_{t+1}\phi_{t+1}]|_{\phi_t=0} &= \mathbb{E}_t\left[(Px_t + \varepsilon_{t+1})(\eta_{t+1} + \zeta_x \varepsilon_{t+1})\right]|_{\phi_t=0} \\
&= Px_t\,\mathbb{E}_t[\eta_{t+1} + \zeta_x\varepsilon_{t+1}] + \mathbb{E}_t[\varepsilon_{t+1}\eta_{t+1}] + \mathbb{E}_t[\varepsilon_{t+1}\varepsilon'_{t+1}]\,\zeta'_x \\
&= \Sigma_\varepsilon \zeta'_x \equiv v_{x\phi} \in \mathbb{R}^{K\times 1}.
\end{aligned}
\tag{A.2}
$$

Combining (A.1) and (A.2) and evaluating expectations yields

$$
\mathbb{E}_t z_{t+1}\Big|_{\phi_t=0} = \Big(a + B_z a + C_x v_{x\phi}\Big) + \Big(B_x P + B_z B_x\Big)x_t + B_z^2 z_{t-1}, \tag{A.3}
$$

where $B_z^2 \equiv B_z B_z$.

Next, we characterize $\frac{\partial z_{t+1}}{\partial \phi_t}$. To do so, note that the processes for x_t and ϕ_t from equations (2) and (3) imply the following auxiliary relations:

$$
\begin{aligned}
\frac{\partial \phi_{t+1}}{\partial \phi_t} &= \rho_\phi, \\
\frac{\partial x_{t+1}}{\partial \phi_t} &= 0, \\
\frac{\partial z_t}{\partial \phi_t} &= d + C_x x_t + C_z z_{t-1},
\end{aligned}
$$

where the last equation follows from the perceived LoM. Let us differentiate z_{t+1}

$$\frac{\partial z_{t+1}}{\partial \phi_t} = B_z \frac{\partial z_t}{\partial \phi_t} + C_x \left(\frac{\partial x_{t+1}}{\partial \phi_t} \phi_{t+1} + x_{t+1} \frac{\partial \phi_{t+1}}{\partial \phi_t} \right) + C_z \left(\frac{\partial z_t}{\partial \phi_t} \phi_{t+1} + z_t \frac{\partial \phi_{t+1}}{\partial \phi_t} \right) + d \frac{\partial \phi_{t+1}}{\partial \phi_t}$$
$$= B_z \left[d + C_x x_t + C_z z_{t-1} \right] + C_x x_{t+1} \rho_\phi + C_z \left(\left[d + C_x x_t + C_z z_{t-1} \right] \phi_{t+1} + z_t \rho_\phi \right) + d \rho_\phi,$$

where the second line uses the auxiliary relations from above. Grouping terms, imposing $\phi_t = 0$, and evaluating at the time-t conditional expectation yields

$$\mathbb{E}_t \left. \frac{\partial z_{t+1}}{\partial \phi_t} \right|_{\phi_t = 0} = \Big(B_z d + C_z \rho_\phi a + d \rho_\phi \Big)$$
$$+ \Big(B_z C_x + C_x \rho_\phi P + C_z \rho_\phi B_x \Big) x_t + \Big(B_z C_z + C_z \rho_\phi B_z \Big) z_{t-1}. \tag{A.4}$$

Substituting expectational terms. Substituting (A.3) and (A.4) into (5) and grouping terms yields

$$\begin{aligned} z_t = {} & \underbrace{F_0 \Big(a + B_z a + C_x v_{x\phi} \Big)}_{=a} \\ & + \underbrace{\Big(F_0 \Big(B_x P + B_z B_x \Big) + G_0 \Big)}_{=B_x} x_t \\ & + \underbrace{\Big(F_0 B_z^2 + H_0 \Big)}_{=B_z} z_{t-1} \\ & + \phi_t \Bigg[\underbrace{F_1 \Big(a + B_z a + C_x v_{x\phi} \Big) + F_0 \Big(B_z d + C_z \rho_\phi a + d \rho_\phi \Big)}_{=d} \\ & \quad + \underbrace{\Big(F_1 \Big(B_x P + B_z B_x \Big) + F_0 \Big(B_z C_x + C_x \rho_\phi P + C_z \rho_\phi B_x \Big) + G_1 \Big)}_{=C_x} x_t \\ & \quad + \underbrace{\Big(F_1 B_z^2 + F_0 \Big(B_z C_z + C_z \rho_\phi B_z \Big) + H_1 \Big)}_{=C_z} z_{t-1} \Bigg] \end{aligned} \tag{A.5}$$

This demonstrates that the functional form of the perceived LoM from (6) is also satisfied by the actual LoM, which proves Proposition 1. □

A.2 Proof of Corollary 1

We derive the state-dependent local projection.

h=0. The LoM based on (6) directly gives

$$z_t = \alpha^0 + \beta^0 \varepsilon_{1t} + \gamma^0 \varepsilon_{1t} \phi_t + \delta^0 \phi_t + u_t, \tag{A.6}$$

with

$$\alpha_0 = a, \tag{A.7}$$

$$\beta_0 = B_x \iota_1, \tag{A.8}$$

$$\gamma_0 = C_x \iota_1, \tag{A.9}$$

$$\delta_0 = d, \tag{A.10}$$

and conformingly defined error term u_t.

h>0. Suppose we have derived the state-dependent local projection representation for $t + h - 1$. Then, we evaluate the LoM at $t + h$:

$$z_{t+h} = a + B_x x_{t+h} + B_z z_{t+h-1} + C_x x_{t+h} \phi_{t+h} + C_z z_{t+h-1} \phi_{t+h} + d\, \phi_{t+h}. \tag{A.11}$$

and substitute z_{t+h-1} to obtain

$$z_{t+h} = a + B_x x_{t+h} + B_z \Big(\alpha_{h-1} + \beta_{h-1} \varepsilon_{1,t} + \gamma_{h-1} \varepsilon_{1,t} \phi_t + \delta_{h-1} \phi_t + u_{t,h-1} \Big) \tag{A.12}$$

$$+ C_x x_{t+h} \phi_{t+h} + C_z \Big(\alpha_{h-1} + \beta_{h-1} \varepsilon_{1,t} + \gamma_{h-1} \varepsilon_{1,t} \phi_t + \delta_{h-1} \phi_t + u_{t,h-1} \Big) \phi_{t+h} + d\, \phi_{t+h}. \tag{A.13}$$

Iterating x_{t+h} and ϕ_{t+h} backwards until time t and using the first vector of the conforming identity matrix, ι_1, and collecting terms for $\alpha^h, \beta^h, \gamma^h$, and δ^h yields the result.

□

A.3 Proof of Corollaries 2 and 3

The proof of Proposition 1 provides a set of equations that characterize the rational expectations solution. Solving these equations yields both Corollaries.
Assumption 1 implies that

$$z_t = F_0 \, \mathbb{E}_t z_{t+1} + G_0 \, x_t + H_0 \, z_{t-1} \tag{A.14}$$

has a unique bounded solution. It is easy to see that this solution is fully characterized by the coefficient matrices B_x and B_z that solve the corresponding equations from (A.5):

$$0 = F_0 B_x P + F_0 B_z B_x + G_0 - B_x \tag{A.15}$$

$$0 = F_0 B_z^2 - B_z + H_0 \tag{A.16}$$

Note that these equations depend only on the model matrices and not on other coefficients from our perceived LoM.
Equation (A.16) is the standard matrix polynomial equation, and the Blanchard-Kahn conditions ensure the right root. This is a well-known problem, and we take its solution, and B_z,

as given subsequently. Next, the solution to (A.15) is

$$\underbrace{\left[\mathbb{I}_{NK} - (\mathbb{I}_K \otimes F_0 B_z) - (P^T \otimes F_0)\right]}_{=\mathcal{M}_{B_x}} vec(B_x) = vec(G_0)$$

$$vec(B_x) = \mathcal{M}_{B_x}^{-1}\, vec(G_0)..$$

The Blanchard-Kahn conditions ensure the existence of the inverse of $\mathcal{M}_{B_x}$.

Lastly, we follow the same steps for the remaining equations implied by (A.5).

Matrix C_z must satisfy

$$0 = F_1 B_z^2 + F_0 B_z C_z + F_0 C_z B_z \rho_\phi + H_1 - C_z. \quad \text{(A.17)}$$

Using the vectorization approach as before, we obtain

$$\underbrace{\left[\mathbb{I}_{N^2} - (\mathbb{I}_N \otimes F_0 B_z) - (B_z^T \otimes F_0)\rho_\phi\right]}_{=\mathcal{M}_{C_z}} vec(C_z) = vec(F_1 B_z^2 + H_1)$$

$$vec(C_z) = \mathcal{M}_{C_z}^{-1}\, vec(F_1 B_z^2 + H_1). \quad \text{(A.18)}$$

Matrix C_x must satisfy

$$0 = F_1 B_x P + F_1 B_z B_x + F_0 B_z C_x + F_0 C_x P \rho_\phi + F_0 C_z B_x \rho_\phi + G_1 - C_x. \quad \text{(A.19)}$$

Using the vectorization approach as before, we obtain

$$\underbrace{\left[\mathbb{I}_{NK} - (\mathbb{I}_K \otimes F_0 B_z) - (P^T \otimes F_0)\rho_\phi\right]}_{=\mathcal{M}_{C_x}} vec(C_x) = vec(F_1 B_x P + F_1 B_z B_x + F_0 C_z B_x \rho_\phi + G_1)$$

$$vec(C_x) = \mathcal{M}_{C_x}^{-1}\, vec(F_1 B_x P + F_1 B_z B_x + F_0 C_z B_x \rho_\phi + G_1). \quad \text{(A.20)}$$

Note that (A.18) and (A.20) provide explicit solutions to C_x and C_z, which only depend on model parameters and the previously determined coefficients B_x and B_z.

Vector a must satisfy

$$\begin{aligned} 0 &= F_0 a + F_0 B_z a + F_0 C_x v_{x\phi} - a \\ \Leftrightarrow a &= \underbrace{\left[\mathbb{I}_N - F_0 - F_0 B_z\right]^{-1}}_{=\mathcal{M}_a^{-1}} F_0 C_x v_{x\phi}. \end{aligned} \tag{A.21}$$

This constitutes an explicit formula for a, given that $B_x,\ B_z,\ C_x$, and C_z have been determined. It also implies an explicit solution to vector d, which must satisfy

$$\begin{aligned} 0 &= F_1 a + F_1 B_z a + F_1 C_x v_{x\phi} + F_0 B_z d + F_0 C_z \rho_\phi a + F_0 d \rho_\phi - d \\ \Leftrightarrow d &= \underbrace{\left[\mathbb{I}_N - F_0 B_z - F_0 \rho_\phi\right]^{-1}}_{=\mathcal{M}_d^{-1}} \left(F_1 a + F_1 B_z a + F_1 C_x v_{x\phi} + F_0 C_z \rho_\phi a\right) \end{aligned} \tag{A.22}$$

Conditional on the linear solution, B_x and B_z, the remaining coefficient matrices a, C_x, C_z, and d are determined by a recursive sequence of linear systems. Each step in this sequence takes the form $\mathcal{M}x = b$, where x is the vectorized coefficient block and $\mathcal{M}, b$ are matrices of known parameters and previously determined coefficients. Moreover, each step consists of solving a system of linear equations with the same number of equations as parameters. This implies that the model admits a unique solution if and only if $\mathcal{M}$ is invertible in each step. Thus, the sequential invertibility of $\mathcal{M}_a, \mathcal{M}_{C_z}, \mathcal{M}_{C_x}$, and $\mathcal{M}_d$ is a necessary and sufficient condition for the existence of a unique solution within the functional class of the perceived LoM, which completes the proof. □

A.4 Proof of Proposition 2

The proof relies on two well-known linear algebra results, which we state first for the convenience of the reader.

Result 1. *Consider the linear matrix equation, also known as the Sylvester equation*

$$\mathcal{A}X - X\mathcal{B} = \mathcal{C},$$

with all symbols denoting conforming matrices. This equation has a unique solution X for any $\mathcal{C}$ if and only if the sets of eigenvalues of $\mathcal{A}$ and $\mathcal{B}$ are disjoint, i.e., $\sigma(\mathcal{A}) \cap \sigma(\mathcal{B}) = \emptyset$.

For a proof, see, e.g., Theorem VII.2.1 in Bhatia (1997).

Result 2. *Let X be a block upper-triangular matrix of the form*

$$X = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{D} \end{bmatrix},$$

with $\mathcal{A}$ and $\mathcal{D}$ being square matrices. Then, the eigenvalues of X are the eigenvalues of its diagonal blocks, i.e., $\sigma(X) = \sigma(\mathcal{A}) \cup \sigma(\mathcal{D})$.

The result follows directly from the fact that $\det(X) = \det(\mathcal{A}) \det(\mathcal{D})$, which is proven in, e.g., Section 6.1.16 of Meyer (2023).

Proof for B_z and B_x. This follows from Assumption 1; see Blanchard and Kahn (1980).

Proof for C_z. Given these auxiliary results, we start the proof using Assumption 3, which allows us to rewrite (16), the equation that pins down C_z, to take the Sylvester form

$$(F_0^{-1} - B_z)C_z - C_z B_z \rho_\phi = F_0^{-1}(F_1 B_z^2 + H_1),$$

conditional on B_z, which follows from Assumption 1. Let $\mathcal{A} = (F_0^{-1} - B_z)$ and $\mathcal{B} = B_z \rho_\phi$. By Result 1, a unique solution C_z exists if and only if $\sigma(\mathcal{A}) \cap \sigma(\mathcal{B}) = \emptyset$.

Further, Assumption 1 implies that all eigenvalues of B_z lie strictly inside the unit circle, which also holds for $\mathcal{B} = B_z \rho_\phi$ as $|\rho_\phi| < 1$, i.e., $|\lambda| < 1 \ \ \forall \lambda \in \sigma(\mathcal{B})$.

Thus, it is sufficient to verify that all eigenvalues of $\mathcal{A} = (F_0^{-1} - B_z)$ lie strictly outside the unit circle. To show this, consider the Blanchard-Kahn form of the nested linear model

$$\begin{bmatrix} z_t \\ z_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ -F_0^{-1}H_0 & F_0^{-1} \end{bmatrix}}_{C} \begin{bmatrix} z_{t-1} \\ z_t \end{bmatrix} + Dx_t.$$

Note that $\sigma(C) = S \cup U$ with $S = \{\lambda_1^S, ..., \lambda_N^S\}$ and $U = \{\lambda_1^U, ..., \lambda_N^U\}$ and $|\lambda_i^S| < 1 < |\lambda_j^U|$ for all i, j, since the Blanchard-Kahn conditions are satisfied. (Remember that N is the length of vector z_t.) As a next step, we use a similarity (eigenvalue preserving) transformation of C to exploit its block upper-triangular structure to compute eigenvalues. Specifically, consider the transformation matrix

$$T = \begin{bmatrix} I & 0 \\ -B_z & I \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} I & 0 \\ B_z & I \end{bmatrix}$$

Since $\det(T) = 1$, T is invertible. We construct the similar matrix $\tilde{C} = TCT^{-1}$:

$$\tilde{C} = \begin{bmatrix} I & 0 \\ -B_z & I \end{bmatrix} \begin{bmatrix} 0 & I \\ -F_0^{-1}H_0 & F_0^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ B_z & I \end{bmatrix} = \begin{bmatrix} B_z & I \\ \underbrace{-F_0^{-1}H_0 + (F_0^{-1} - B_z)B_z}_{R_{21}} & F_0^{-1} - B_z \end{bmatrix}$$

To obtain that $\tilde{C}$ is a block upper triangular matrix, we need to show $R_{21} = \mathbf{0}$. To demonstrate this, we recall the matrix equation defining B_z, which is

$$F_0 B_z^2 - B_z + H_0 = 0 \quad \Leftrightarrow \quad F_0^{-1}H_0 = F_0^{-1}B_z - B_z^2,$$

given invertibility of F_0. Substituting this yields $R_{21} = \mathbf{0}$. This implies $\tilde{C}$ is block upper triangular and Result 2 applies so that $\sigma(\tilde{C}) = \sigma(B_z) \cup \sigma(F_0^{-1} - B_z)$. By similarity, $\sigma(\tilde{C}) = \sigma(C) = S \cup U$. We know $\sigma(B_z) = S$ due to the Blanchard-Kahn conditions. Therefore, by elimination, it must be that $\sigma(F_0^{-1} - B_z) = U$. Consequently, all eigenvalues of $\mathcal{A}$ have modulus strictly greater than unity. Since we have already shown that all eigenvalues of $\mathcal{B}$ are strictly inside the unit circle, the existence of a unique matrix C_z solving the perceived and actual LoM follows from Result 1.

Proof for C_x. We must show the same result for matrix C_x. To this end, we rewrite (17) in Sylvester form

$$(F_0^{-1} - B_z)C_x - C_x\rho_\phi P = F_0^{-1}F_1\Big(B_xP + B_zB_x\Big) + C_z\rho_\phi B_x + F_0^{-1}G_1, \tag{A.23}$$

relying on invertibility of F_0, i.e., Assumption 3. Applying the same notation as before, it is easy to see that all eigenvalues of $\mathcal{B} = \rho_\phi P$ are strictly inside the unit circle and that $\mathcal{A} = F_0^{-1} - B_z$ is identical to the corresponding matrix for C_z. Thus, applying the same steps as for C_z concludes this step.

Proof for a. From Equation 18, we obtain

$$((F_0^{-1} - B_z) - \mathbb{I}_N)a = C_x v_{x\phi} \tag{A.24}$$

with $v_{x\phi} = \Sigma_\varepsilon \zeta_x'$. A unique solution exists when $((F_0^{-1} - B_z) - \mathbb{I}_N)$ is invertible. As shown above, all eigenvalues of $(F_0^{-1} - B_z)$ are strictly outside the unit circle. It follows that for all $\lambda \in \sigma((F_0^{-1} - B_z) - \mathbb{I}_N)$, we have $|\lambda - 1| \geq |\lambda| - 1 > 0$ from the triangle inequality and the result for a follows.

Proof for d. From Equation 19, we obtain

$$((F_0^{-1} - B_z) - \mathbb{I}_N\rho_\phi)d = F_0^{-1}F_1\Big(a + B_za + C_xv_{x\phi}\Big) + C_z\rho_\phi a. \tag{A.25}$$

Applying the same arguments as for a to the left-hand side matrix completes this step since $|\rho_\phi| < 1$.

We have shown that there exists a unique solution for each matrix of the perceived and actual LoM, which concludes the proof. □